8.5 Midsegment of a Trapezoid

MATERIALS • graphing calculator or computer

QUESTION What are the properties of the midsegment of a trapezoid?

You can use geometry drawing software to investigate properties of trapezoids.

EXPLORE Draw a trapezoid and its midsegment

STEP 1 Draw parallel lines Draw \( \overline{AB} \). Draw a point \( C \) not on \( \overline{AB} \) and construct a line parallel to \( \overline{AB} \) through point \( C \).

STEP 2 Draw trapezoid Construct a point \( D \) on the same line as point \( C \). Then draw \( \overline{AD} \) and \( \overline{BC} \) so that the segments are not parallel. Draw \( \overline{AB} \) and \( \overline{DC} \). Quadrilateral \( ABCD \) is called a trapezoid. A trapezoid is a quadrilateral with exactly one pair of parallel sides.

STEP 3 Draw midsegment Construct the midpoints of \( \overline{AD} \) and \( \overline{BC} \). Label the points \( E \) and \( F \). Draw \( \overline{EF} \). \( \overline{EF} \) is called a midsegment of trapezoid \( ABCD \). The midsegment of a trapezoid connects the midpoints of its nonparallel sides.

STEP 4 Measure lengths Measure \( AB \), \( DC \), and \( EF \).

STEP 5 Compare lengths The average of \( AB \) and \( DC \) is \( \frac{AB + DC}{2} \). Calculate and compare this average to \( EF \). What do you notice? Drag point \( A \) or point \( B \) to change the shape of trapezoid \( ABCD \). Do not allow \( \overline{AD} \) to intersect \( \overline{BC} \). What do you notice about \( EF \) and \( \frac{AB + DC}{2} \)?

DRAW CONCLUSIONS Use your observations to complete these exercises

1. Make a conjecture about the length of the midsegment of a trapezoid.

2. The midsegment of a trapezoid is parallel to the two parallel sides of the trapezoid. What measurements could you make to show that the midsegment in the Explore is parallel to \( \overline{AB} \) and \( \overline{CD} \)? Explain.

3. In Lesson 5.1 (page 295), you learned a theorem about the midsegment of a triangle. How is the midsegment of a trapezoid similar to the midsegment of a triangle? How is it different?
8.5 Use Properties of Trapezoids and Kites

Key Vocabulary
- trapezoid
- bases, base angles, legs
- isosceles trapezoid
- midsegment of a trapezoid
- kite

A trapezoid is a quadrilateral with exactly one pair of parallel sides. The parallel sides are the bases.

A trapezoid has two pairs of base angles. For example, in trapezoid $ABCD$, $\angle A$ and $\angle D$ are one pair of base angles, and $\angle B$ and $\angle C$ are the second pair. The nonparallel sides are the legs of the trapezoid.

**Example 1** Use a coordinate plane

Show that $QRST$ is a trapezoid.

**Solution**

Compare the slopes of opposite sides.

Slope of $RS = \frac{4 - 3}{2 - 0} = \frac{1}{2}$

Slope of $OT = \frac{2 - 0}{4 - 0} = \frac{2}{4} = \frac{1}{2}$

The slopes of $RS$ and $OT$ are the same, so $RS \parallel OT$.

Slope of $ST = \frac{2 - 4}{4 - 2} = \frac{-2}{2} = -1$

Slope of $OR = \frac{3 - 0}{0 - 0} = \frac{3}{0}$, which is undefined.

The slopes of $ST$ and $OR$ are not the same, so $ST$ is not parallel to $OR$.

- Because quadrilateral $QRST$ has exactly one pair of parallel sides, it is a trapezoid.

**Guided Practice** for Example 1

1. **WHAT IF?** In Example 1, suppose the coordinates of point $S$ are $(4, 5)$. What type of quadrilateral is $QRST$? **Explain.**

2. In Example 1, which of the interior angles of quadrilateral $QRST$ are supplementary angles? **Explain** your reasoning.
**THEOREMS**

**Theorem 8.14**
If a trapezoid is isosceles, then each pair of base angles is congruent.

If trapezoid $ABCD$ is isosceles, then $\angle A \cong \angle D$ and $\angle B \cong \angle C$.

*Proof:* Ex. 37, p. 548

**Theorem 8.15**
If a trapezoid has a pair of congruent base angles, then it is an isosceles trapezoid.

If $\angle A \cong \angle D$ (or if $\angle B \cong \angle C$), then trapezoid $ABCD$ is isosceles.

*Proof:* Ex. 38, p. 548

**Theorem 8.16**
A trapezoid is isosceles if and only if its diagonals are congruent.

Trapezoid $ABCD$ is isosceles if and only if $AC \cong BD$.

*Proof:* Exs. 39 and 43, p. 549

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**Example 2** Use properties of isosceles trapezoids

**Arch** The stone above the arch in the diagram is an isosceles trapezoid. Find $m\angle K$, $m\angle M$, and $m\angle J$.

**Solution**

**Step 1** Find $m\angle K$. $JKLM$ is an isosceles trapezoid, so $\angle K$ and $\angle L$ are congruent base angles, and $m\angle K = m\angle L = 85^\circ$.

**Step 2** Find $m\angle M$. Because $\angle L$ and $\angle M$ are consecutive interior angles formed by $\overline{LM}$ intersecting two parallel lines, they are supplementary. So, $m\angle M = 180^\circ - 85^\circ = 95^\circ$.

**Step 3** Find $m\angle J$. Because $\angle J$ and $\angle M$ are a pair of base angles, they are congruent, and $m\angle J = m\angle M = 95^\circ$.

So, $m\angle J = 95^\circ$, $m\angle K = 85^\circ$, and $m\angle M = 95^\circ$. 

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**ISOSCELES TRAPEZOIDS** If the legs of a trapezoid are congruent, then the trapezoid is an **isosceles trapezoid**.
**MIDSEGMENTS** Recall that a midsegment of a triangle is a segment that connects the midpoints of two sides of the triangle. The **midsegment of a trapezoid** is the segment that connects the midpoints of its legs.

The theorem below is similar to the Midsegment Theorem for Triangles.

**THEOREM**

**Theorem 8.17 Midsegment Theorem for Trapezoids**

The midsegment of a trapezoid is parallel to each base and its length is one half the sum of the lengths of the bases.

If \( MN \) is the midsegment of trapezoid \( ABCD \), then

\[
MN \parallel AB, MN \parallel DC, \text{ and } MN = \frac{1}{2}(AB + CD).
\]

*Justification:* Ex. 40, p. 549
*Proof:* p. 937

**Example 3** Use the midsegment of a trapezoid

In the diagram, \( MN \) is the midsegment of trapezoid \( PQRS \). Find \( MN \).

**Solution**

Use Theorem 8.17 to find \( MN \).

\[
MN = \frac{1}{2}(PQ + SR) \quad \text{Apply Theorem 8.17.}
\]

\[
= \frac{1}{2}(12 + 28) \quad \text{Substitute 12 for } PQ \text{ and 28 for } XR.
\]

\[
= 20 \quad \text{Simplify.}
\]

\( \triangleright \) The length \( MN \) is 20 inches.

**Guided Practice** for Examples 2 and 3

In Exercises 3 and 4, use the diagram of trapezoid \( EFGH \).

3. If \( EG = FH \), is trapezoid \( EFGH \) isosceles? Explain.

4. If \( m\angle HEG = 70^\circ \) and \( m\angle FGH = 110^\circ \), is trapezoid \( EFGH \) isosceles? Explain.

5. In trapezoid \( JKLM \), \( \angle J \) and \( \angle M \) are right angles, and \( JK = 9 \) cm. The length of the midsegment \( \overline{NP} \) of trapezoid \( JKLM \) is 12 cm. Sketch trapezoid \( JKLM \) and its midsegment. Find \( ML \). Explain your reasoning.
**Kites** A kite is a quadrilateral that has two pairs of consecutive congruent sides, but opposite sides are not congruent.

**Theorems**

**Theorem 8.18**

If a quadrilateral is a kite, then its diagonals are perpendicular.

If quadrilateral $ABCD$ is a kite, then $\overline{AC} \perp \overline{BD}$.

*Proof:* Ex. 41, p. 549

**Theorem 8.19**

If a quadrilateral is a kite, then exactly one pair of opposite angles are congruent.

If quadrilateral $ABCD$ is a kite and $BC \cong BA$, then $\angle A \equiv \angle C$ and $\angle B \equiv \angle D$.

*Proof:* Ex. 42, p. 549

**Example 4**

**Apply Theorem 8.19**

Find $m\angle D$ in the kite shown at the right.

**Solution**

By Theorem 8.19, $DEFG$ has exactly one pair of congruent opposite angles. Because $\angle E \neq \angle G$, $\angle D$ and $\angle F$ must be congruent. So, $m\angle D = m\angle F$.

Write and solve an equation to find $m\angle D$.

\[
m\angle D + m\angle F + 124^\circ + 80^\circ = 360^\circ
\]

\[
m\angle D + m\angle D + 124^\circ + 80^\circ = 360^\circ
\]

\[
2(m\angle D) + 204^\circ = 360^\circ
\]

\[
\frac{2(m\angle D) + 204^\circ}{2} = \frac{360^\circ}{2}
\]

\[
m\angle D = 78^\circ
\]

**Guided Practice**

6. In a kite, the measures of the angles are $3x^\circ, 75^\circ, 90^\circ$, and $120^\circ$. Find the value of $x$. What are the measures of the angles that are congruent?
1. **VOCABULARY** In trapezoid $PQRS$, $PQ \parallel RS$. Sketch $PQRS$ and identify its bases and its legs.

2. **WRITING** Describe the differences between a kite and a trapezoid.

**COORDINATE PLANE** Points $A$, $B$, $C$, and $D$ are the vertices of a quadrilateral. Determine whether $ABCD$ is a trapezoid.

3. $A(0, 4), B(4, 4), C(8, 2), D(2, 1)$
4. $A(-5, 0), B(2, 3), C(3, 1), D(-2, -2)$
5. $A(2, 1), B(6, 1), C(3, -3), D(-1, -4)$
6. $A(-3, 3), B(-1, 1), C(1, -4), D(-3, 0)$

**FINDING ANGLE MEASURES** Find $m\angle J$, $m\angle L$, and $m\angle M$.

7. $\triangle JKL$
8. $\triangle JKL$
9. $\triangle JKL$

**REASONING** Determine whether the quadrilateral is a trapezoid. Explain.

10. $A \quad B \quad D \quad C$
11. $E \quad F \quad G \quad H$
12. $J \quad K \quad M \quad N$

**FINDING MIDSEGMENTS** Find the length of the midsegment of the trapezoid.

13. $\triangle 18 \quad 10$
14. $\triangle 21 \quad 25$
15. $\triangle 57 \quad 76$

16. **MULTIPLE CHOICE** Which statement is not always true?
   - A. The base angles of an isosceles trapezoid are congruent.
   - B. The midsegment of a trapezoid is parallel to the bases.
   - C. The bases of a trapezoid are parallel.
   - D. The legs of a trapezoid are congruent.

17. **ERROR ANALYSIS** Describe and correct the error made in finding $m\angle A$.Opposite angles of a kite are congruent, so $m\angle A = 50^\circ$. 

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**EXAMPLES**

**1 and 2** on pp. 542–543 for Exs. 3–12

**EXAMPLE 3** on p. 544 for Exs. 13–16

**EXAMPLE 4** on p. 545 for Exs. 17–20

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**HOMEWORK KEY**

- = WORKED-OUT SOLUTIONS on p. WS1 for Exs. 11, 19, and 35
- = STANDARDIZED TEST PRACTICE Exs. 2, 16, 28, 31, and 36

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ANGLES OF KITES  $EFGH$ is a kite. Find $m \angle G$.

18. [Diagram of EFGH with angles 100°, 40°, 100°, and 40°]

19. [Diagram of EFGH with angles 60°, 110°, 60°, and 110°]

20. [Diagram of EFGH with angles 150°, 30°, 150°, and 30°]

DIAGONALS OF KITES  Use Theorem 8.18 and the Pythagorean Theorem to find the side lengths of the kite. Write the lengths in simplest radical form.

21. [Diagram of XWYZ with sides 3, 5, 3, and 5]

22. [Diagram of XWYZ with sides 6, 12, 6, and 12]

23. [Diagram of XWYZ with sides 10, 5, 10, and 5]

24. ERROR ANALYSIS  In trapezoid $ABCD$, $MN$ is the midsegment. Describe and correct the error made in finding $DC$.

25. [Diagram of trapezoid with sides 10, 7, 2, and x]

26. [Diagram of trapezoid with sides 12.5, 15, 3x + 1, and 3x]

27. [Diagram of trapezoid with sides 18.7, 12x - 1.7, 5x, and x]

28. ★ SHORT RESPONSE  The points $M(-3, 5), N(-1, 5), P(3, -1),$ and $Q(-5, -1)$ form the vertices of a trapezoid. Draw $MNPQ$ and find $MP$ and $NQ$. What do your results tell you about the trapezoid? Explain.

29. DRAWING  In trapezoid $JKLM$, $JK \parallel LM$ and $JK = 17$. The midsegment of $JKLM$ is $XY$, and $XY = 37$. Sketch $JKLM$ and its midsegment. Then find $LM$.

30. RATIOS  The ratio of the lengths of the bases of a trapezoid is $1:3$. The length of the midsegment is 24. Find the lengths of the bases.

31. ★ MULTIPLE CHOICE  In trapezoid $PQRS$, $PQ \parallel RS$ and $MN$ is the midsegment of $PQRS$. If $RS = 5 \cdot PQ$, what is the ratio of $MN$ to $RS$?

   A) 3:5  B) 5:3  C) 2:1  D) 3:1

32. CHALLENGE  The figure shown at the right is a trapezoid with its midsegment. Find all the possible values of $x$. What is the length of the midsegment? Explain. (The figure may not be drawn to scale.)

33. REASONING  Explain why a kite and a general quadrilateral are the only quadrilaterals that can be concave.
34. **FURNITURE** In the photograph of a chest of drawers, \( HC \) is the midsegment of trapezoid \( ABDG \), \( GD \) is the midsegment of trapezoid \( HCEF \), \( AB = 13.9 \) centimeters, and \( GD = 50.5 \) centimeters. Find \( HC \). Then find \( FE \).

35. **GRAPHIC DESIGN** You design a logo in the shape of a convex kite. The measure of one angle of the kite is \( 90^\circ \). The measure of another angle is \( 30^\circ \). Sketch a kite that matches this description. Give the measures of all the angles and mark any congruent sides.

36. **EXTENDED RESPONSE** The bridge below is designed to fold up into an octagon shape. The diagram shows a section of the bridge.

   a. Classify the quadrilaterals shown in the diagram.
   
   b. As the bridge folds up, what happens to the length of \( BF \)? What happens to \( m\angle BAF \), \( m\angle ABC \), \( m\angle BCF \), and \( m\angle CFA \)?
   
   c. Given \( m\angle CFE = 65^\circ \), find \( m\angle DEF \), \( m\angle FCD \), and \( m\angle CDE \). Explain.

37. **PROVING THEOREM 8.14** Use the diagram and the auxiliary segment to prove Theorem 8.14. In the diagram, \( EC \) is drawn parallel to \( AB \).

   **GIVEN** \( \triangle ABCD \) is an isosceles trapezoid, \( BC \parallel AD \)

   **PROVE** \( \angle A \equiv \angle D, \angle B \equiv \angle BCD \)

   **Hint:** Find a way to show that \( \triangle ECD \) is an isosceles triangle.

38. **PROVING THEOREM 8.15** Use the diagram and the auxiliary segment to prove Theorem 8.15. In the diagram, \( FG \) is drawn parallel to \( EH \).

   **GIVEN** \( EFGH \) is a trapezoid, \( FG \parallel EH, \angle E \equiv \angle H \)

   **PROVE** \( EFGH \) is an isosceles trapezoid.

   **Hint:** Find a way to show that \( \triangle JGH \) is an isosceles triangle.
39. **PROVING THEOREM 8.16** Prove part of Theorem 8.16.

**GIVEN** ▶ \(JKLM\) is an isosceles trapezoid.
\(KL \parallel JM, JK \cong LM\)

**PROVE** ▶ \(JL \cong KM\)

40. **REASONING** In the diagram below, \(BG\) is the midsegment of \(\triangle ACD\) and \(GE\) is the midsegment of \(\triangle ADF\). Explain why the midsegment of trapezoid \(ACDF\) is parallel to each base and why its length is one half the sum of the lengths of the bases.

41. **PROVING THEOREM 8.18** Prove Theorem 8.18.

**GIVEN** ▶ \(ABCD\) is a kite.
\(AB \cong CB, AD \cong CD\)

**PROVE** ▶ \(AC \perp BD\)

42. **PROVING THEOREM 8.19** Write a paragraph proof of Theorem 8.19.

**GIVEN** ▶ \(EFGH\) is a kite.
\(EF \cong GF, EH \cong GH\)

**PROVE** ▶ \(\angle E \cong \angle G, \angle F \cong \angle H\)

**Plan for Proof** First show that \(\angle E \cong \angle G\). Then use an indirect argument to show that \(\angle F \cong \angle H\): If \(\angle F \cong \angle H\), then \(EFGH\) is a parallelogram. But opposite sides of a parallelogram are congruent. This result contradicts the definition of a kite.

43. **CHALLENGE** In Exercise 39, you proved that part of Theorem 8.16 is true. Write the other part of Theorem 8.16 as a conditional statement. Then prove that the statement is true.

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**MIXED REVIEW**

44. Place a right triangle in a coordinate plane in a way that is convenient for finding side lengths. Assign coordinates to each vertex. (p. 295)

Use the diagram to complete the proportion. (p. 449)

45. \(\frac{AB}{AC} = \frac{?}{AB}\)

46. \(\frac{AB}{BC} = \frac{BD}{?}\)

Three of the vertices of \(\square ABCD\) are given. Find the coordinates of point \(D\). Show your method. (p. 522)

47. \(A(-1, -2), B(4, -2), C(6, 2), D(x, y)\)

48. \(A(1, 4), B(0, 1), C(4, 1), D(x, y)\)

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**EXTRA PRACTICE** for Lesson 8.5, p. 911  
**ONLINE QUIZ** at classzone.com
**Extension**

*Use after Lesson 8.5*

**Draw Three-Dimensional Figures**

**Goal**

Create isometric drawings and orthographic projections of three-dimensional figures.

**Technical drawings** are drawings that show different viewpoints of an object. Engineers and architects create technical drawings of products and buildings before actually constructing the actual objects.

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**Example 1**

**Draw a rectangular box**

**Solution**

**Step 1** Draw the bases. They are rectangular, but you need to draw them tilted.

**Step 2** Connect the bases using vertical lines.

**Step 3** Erase parts of the hidden edges so that they are dashed lines.

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**Example 2**

**Create an isometric drawing**

Create an isometric drawing of the rectangular box in Example 1.

**Solution**

**Step 1** Draw three axes on isometric dot paper.

**Step 2** Draw the box so that the edges of the box are parallel to the three axes.

**Step 3** Add depth to the drawing by using different shading for the front, top, and sides.
ANOTHER VIEW  Technical drawings may also include an orthographic projection. An orthographic projection is a two-dimensional drawing of the front, top, and side views of an object. The interior lines in these two-dimensional drawings represent edges of the object.

EXAMPLE 3  Create an orthographic projection

Create an orthographic projection of the solid.

Solution

On graph paper, draw the front, top, and side views of the solid.

VISUAL REASONING

In this Extension, you can think of the solids as being constructed from cubes. You can assume there are no cubes hidden from view except those needed to support the visible ones.

PRACTICE

DRAWING BOXES  Draw a box with the indicated base.
1. Equilateral triangle  2. Regular hexagon  3. Square

DRAWING SOLIDS  Create an isometric drawing of the solid. Then create an orthographic projection of the solid.
4. 5. 6.
7. 8. 9.

CREATING ISOMETRIC DRAWINGS  Create an isometric drawing of the orthographic projection.
10. 11. 12.