11.4 Circumference and Arc Length

Before You found the circumference of a circle.

Now You will find arc lengths and other measures.

Why? So you can find a running distance, as in Example 5.

Key Vocabulary
- **circumference**
- **arc length**
- **radius,** *p. 651*
- **diameter,** *p. 651*
- **measure of an arc,** *p. 659*

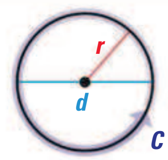
The **circumference** of a circle is the distance around the circle. For all circles, the ratio of the circumference to the diameter is the same. This ratio is known as π, or *pi*. In Chapter 1, you used 3.14 to approximate the value of π. Throughout this chapter, you should use the π key on a calculator, then round to the hundredths place unless instructed otherwise.

THEOREM *For Your Notebook*

THEOREM 11.8 Circumference of a Circle

The circumference C of a circle is $C = \pi d$ or $C = 2\pi r$, where d is the diameter of the circle and r is the radius of the circle.

Justification: Ex. 2, p. 769



$C = \pi d = 2\pi r$

EXAMPLE 1 Use the formula for circumference

Find the indicated measure.

a. Circumference of a circle with radius 9 centimeters

b. Radius of a circle with circumference 26 meters

Solution

a.
$C = 2\pi r$ Write circumference formula.
$= 2 \cdot \pi \cdot 9$ Substitute 9 for r.
$= 18\pi$ Simplify.
≈ 56.55 Use a calculator.

▶ The circumference is about 56.55 centimeters.

b.
$C = 2\pi r$ Write circumference formula.
$26 = 2\pi r$ Substitute 26 for C.
$\frac{26}{2\pi} = r$ Divide each side by 2π.
$4.14 \approx r$ Use a calculator.

▶ The radius is about 4.14 meters.

ANOTHER WAY
You can give an exact measure in terms of π. In Example 1, part (a), the exact circumference is 18π. The exact radius in Example 1, part (b) is $\frac{26}{2\pi}$, or $\frac{13}{\pi}$.

EXAMPLE 2 Use circumference to find distance traveled

TIRE REVOLUTIONS The dimensions of a car tire are shown at the right. To the nearest foot, how far does the tire travel when it makes 15 revolutions?

Solution

STEP 1 **Find** the diameter of the tire.

$d = 15 + 2(5.5) = 26$ in.

STEP 2 **Find** the circumference of the tire.

$C = \pi d = \pi(26) \approx 81.68$ in.

STEP 3 **Find** the distance the tire travels in 15 revolutions. In one revolution, the tire travels a distance equal to its circumference. In 15 revolutions, the tire travels a distance equal to 15 times its circumference.

Distance traveled = Number of revolutions • Circumference

$\approx 15 \cdot 81.68$ in.

$= 1225.2$ in.

AVOID ERRORS
Always pay attention to units. In Example 2, you need to convert units to get a correct answer.

STEP 4 **Use** unit analysis. Change 1225.2 inches to feet.

$$1225.2 \cancel{\text{in.}} \cdot \frac{1 \text{ ft}}{12 \cancel{\text{in.}}} = 102.1 \text{ ft}$$

▶ The tire travels approximately 102 feet.

✓ GUIDED PRACTICE for Examples 1 and 2

1. Find the circumference of a circle with diameter 5 inches. Find the diameter of a circle with circumference 17 feet.

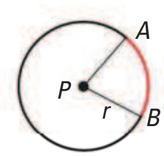
2. A car tire has a diameter of 28 inches. How many revolutions does the tire make while traveling 500 feet?

ARC LENGTH An **arc length** is a portion of the circumference of a circle. You can use the measure of the arc (in degrees) to find its length (in linear units).

COROLLARY *For Your Notebook*

ARC LENGTH COROLLARY

In a circle, the ratio of the length of a given arc to the circumference is equal to the ratio of the measure of the arc to 360°.



$$\frac{\text{Arc length of } \overset{\frown}{AB}}{2\pi r} = \frac{m\overset{\frown}{AB}}{360^\circ}, \text{ or Arc length of } \overset{\frown}{AB} = \frac{m\overset{\frown}{AB}}{360^\circ} \cdot 2\pi r$$

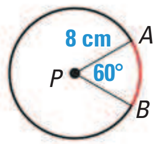
EXAMPLE 3 Find arc lengths

INTERPRET DIAGRAMS

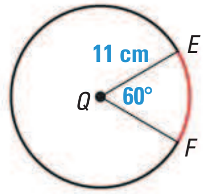
In Example 3, $\overparen{AB}$ and $\overparen{EF}$ have the same measure. However, they have different lengths because they are in circles with different circumferences.

Find the length of each red arc.

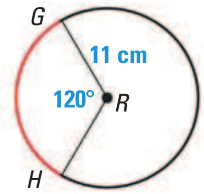
a.



b.



c.



Solution

a. Arc length of $\overparen{AB} = \frac{60^\circ}{360^\circ} \cdot 2\pi(8) \approx 8.38$ centimeters

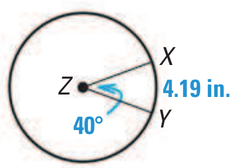
b. Arc length of $\overparen{EF} = \frac{60^\circ}{360^\circ} \cdot 2\pi(11) \approx 11.52$ centimeters

c. Arc length of $\overparen{GH} = \frac{120^\circ}{360^\circ} \cdot 2\pi(11) \approx 23.04$ centimeters

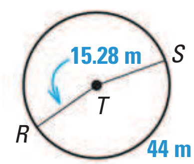
EXAMPLE 4 Use arc lengths to find measures

Find the indicated measure.

a. Circumference C of $\odot Z$



b. $m\overparen{RS}$



Solution

a.

$$\frac{\text{Arc length of } \overparen{XY}}{C} = \frac{m\overparen{XY}}{360^\circ}$$

$$\frac{4.19}{C} = \frac{40^\circ}{360^\circ}$$

$$\frac{4.19}{C} = \frac{1}{9}$$

▶ $37.71 = C$

b.

$$\frac{\text{Arc length of } \overparen{RS}}{2\pi r} = \frac{m\overparen{RS}}{360^\circ}$$

$$\frac{44}{2\pi(15.28)} = \frac{m\overparen{RS}}{360^\circ}$$

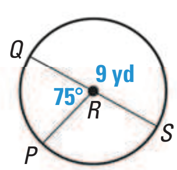
$$360^\circ \cdot \frac{44}{2\pi(15.28)} = m\overparen{RS}$$

▶ $165^\circ \approx m\overparen{RS}$

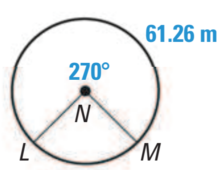
✓ GUIDED PRACTICE for Examples 3 and 4

Find the indicated measure.

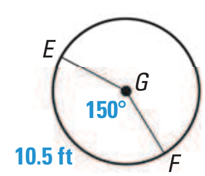
3. Length of $\overparen{PQ}$



4. Circumference of $\odot N$

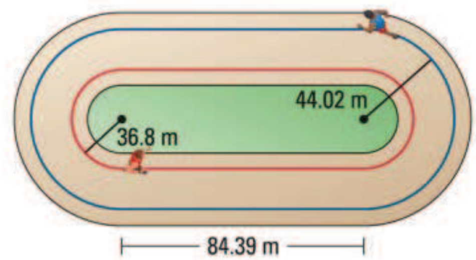


5. Radius of $\odot G$



EXAMPLE 5 Use arc length to find distances

TRACK The curves at the ends of the track shown are 180° arcs of circles. The radius of the arc for a runner on the red path shown is 36.8 meters. About how far does this runner travel to go once around the track? Round to the nearest tenth of a meter.



Solution

USE FORMULAS
The arc length of a semicircle is half the circumference of the circle with the same radius. So, the arc length of a semicircle is $\frac{1}{2} \cdot 2\pi r$, or πr.

The path of a runner is made of two straight sections and two semicircles. To find the total distance, find the sum of the lengths of each part.

Distance = 2 • Length of each straight section + 2 • Length of each semicircle

$$= 2(84.39) + 2 \cdot \left(\frac{1}{2} \cdot 2\pi \cdot 36.8\right)$$

$$\approx 400.0 \text{ meters}$$

▶ The runner on the red path travels about 400 meters.

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✓ GUIDED PRACTICE for Example 5

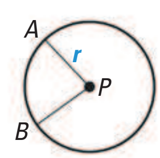
6. In Example 5, the radius of the arc for a runner on the blue path is 44.02 meters, as shown in the diagram. About how far does this runner travel to go once around the track? Round to the nearest tenth of a meter.

11.4 EXERCISES

HOMEWORK KEY
○ = **WORKED-OUT SOLUTIONS** on p. WS1 for Exs. 23, 25, and 35
★ = **STANDARDIZED TEST PRACTICE** Exs. 2, 31, 32, and 38

SKILL PRACTICE

In Exercises 1 and 2, refer to the diagram of $\odot P$ shown.



1. **VOCABULARY** Copy and complete the equation: $\frac{?}{2\pi r} = \frac{m\widehat{AB}}{?}$.

2. ★ **WRITING** *Describe* the difference between the *arc measure* and the *arc length* of $\widehat{AB}$.

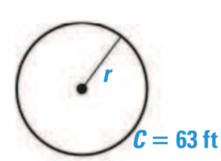
EXAMPLE 1
on p. 746 for Exs. 3–7

USING CIRCUMFERENCE **Use the diagram to find the indicated measure.**

3. Find the circumference.

4. Find the circumference.

5. Find the radius.

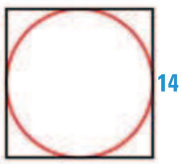


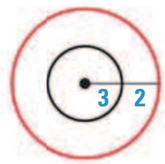
FINDING EXACT MEASURES **Find the indicated measure.**

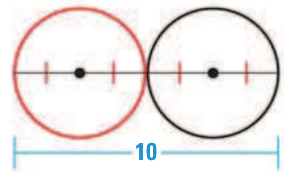
6. The exact circumference of a circle with diameter 5 inches
7. The exact radius of a circle with circumference 28π meters

EXAMPLE 2
on p. 747
for Exs. 8–10

FINDING CIRCUMFERENCE **Find the circumference of the red circle.**

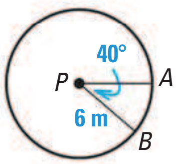
8. 

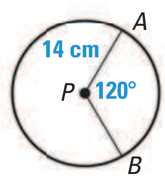
9. 

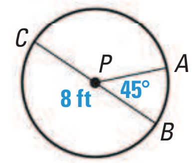
10. 

EXAMPLE 3
on p. 748
for Exs. 11–20

FINDING ARC LENGTHS **Find the length of $\overset{\frown}{AB}$.**

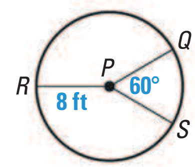
11. 

12. 

13. 

14. **ERROR ANALYSIS** A student says that two arcs from different circles have the same arc length if their central angles have the same measure. *Explain* the error in the student's reasoning.

FINDING MEASURES **In $\odot P$ shown at the right, $\angle QPR \cong \angle RPS$. Find the indicated measure.**

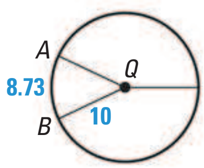


15. $m\overset{\frown}{QRS}$
16. Length of $\overset{\frown}{QRS}$
17. $m\overset{\frown}{QR}$
18. $m\overset{\frown}{RSQ}$
19. Length of $\overset{\frown}{QR}$
20. Length of $\overset{\frown}{RSQ}$

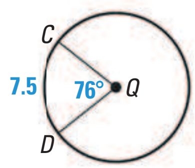
EXAMPLE 4
on p. 748
for Exs. 21–23

USING ARC LENGTH **Find the indicated measure.**

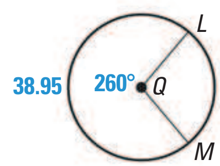
21. $m\overset{\frown}{AB}$



22. Circumference of $\odot Q$

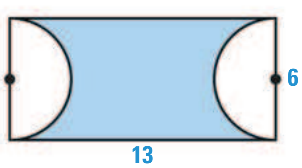


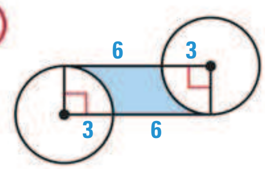
23. Radius of $\odot Q$



EXAMPLE 5
on p. 749
for Exs. 24–25

FINDING PERIMETERS **Find the perimeter of the shaded region.**

24. 

25. 

COORDINATE GEOMETRY **The equation of a circle is given. Find the circumference of the circle. Write the circumference in terms of π.**

26. $x^2 + y^2 = 16$
27. $(x + 2)^2 + (y - 3)^2 = 9$
28. $x^2 + y^2 = 18$

29. *xy* **ALGEBRA** Solve the formula $C = 2\pi r$ for r. Solve the formula $C = \pi d$ for d. Use the rewritten formulas to find r and d when $C = 26\pi$.

= **WORKED-OUT SOLUTIONS** on p. WS1

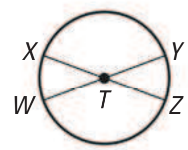
★ = **STANDARDIZED TEST PRACTICE**

30. **FINDING VALUES** In the table below, $\widehat{AB}$ refers to the arc of a circle. Copy and complete the table.

Radius	?	2	0.8	4.2	?	$4\sqrt{2}$
$m\widehat{AB}$	45°	60°	?	183°	90°	?
Length of $\widehat{AB}$	4	?	0.3	?	3.22	2.86

31. ★ **SHORT RESPONSE** Suppose $\widehat{EF}$ is an arc on a circle with radius r. Let $x°$ be the measure of $\widehat{EF}$. *Describe* the effect on the length of $\widehat{EF}$ if you (a) double the radius of the circle, and (b) double the measure of $\widehat{EF}$.

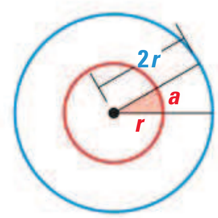
32. ★ **MULTIPLE CHOICE** In the diagram, $\overline{WY}$ and $\overline{XZ}$ are diameters of $\odot T$, and $WY = XZ = 6$. If $m\widehat{XY} = 140°$, what is the length of $\widehat{YZ}$?



Ⓐ $\frac{2}{3}\pi$ Ⓑ $\frac{4}{3}\pi$ Ⓒ 6π Ⓓ 4π

33. **CHALLENGE** Find the circumference of a circle inscribed in a rhombus with diagonals that are 12 centimeters and 16 centimeters long. *Explain*.

34. **FINDING CIRCUMFERENCE** In the diagram, the measure of the shaded red angle is 30°. The arc length a is 2. *Explain* how to find the circumference of the blue circle without finding the radius of either the red or the blue circles.



PROBLEM SOLVING

35. **TREES** A group of students wants to find the diameter of the trunk of a young sequoia tree. The students wrap a rope around the tree trunk, then measure the length of rope needed to wrap one time around the trunk. This length is 21 feet 8 inches. *Explain* how they can use this length to estimate the diameter of the tree trunk to the nearest half foot.

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36. **INSCRIBED SQUARE** A square with side length 6 units is inscribed in a circle so that all four vertices are on the circle. Draw a sketch to represent this problem. Find the circumference of the circle.

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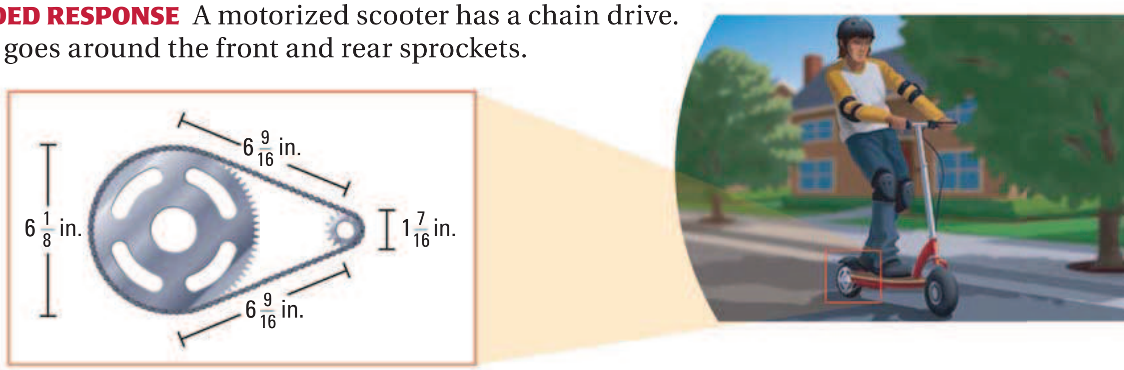
EXAMPLE 2
on p. 747
for Ex. 37

37. **MEASURING WHEEL** As shown, a measuring wheel is used to calculate the length of a path. The diameter of the wheel is 8 inches. The wheel rotates 87 times along the length of the path. About how long is the path?

38. ★ **EXTENDED RESPONSE** A motorized scooter has a chain drive. The chain goes around the front and rear sprockets.

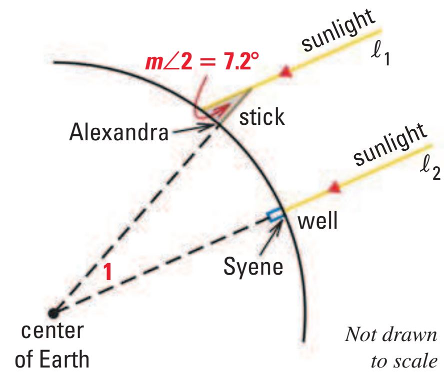


a. About how long is the chain? *Explain*.

b. Each sprocket has teeth that grip the chain. There are 76 teeth on the larger sprocket, and 15 teeth on the smaller sprocket. About how many teeth are gripping the chain at any given time? *Explain*.

39. **SCIENCE** Over 2000 years ago, the Greek scholar Eratosthenes estimated Earth's circumference by assuming that the Sun's rays are parallel. He chose a day when the Sun shone straight down into a well in the city of Syene. At noon, he measured the angle the Sun's rays made with a vertical stick in the city of Alexandria. Eratosthenes assumed that the distance from Syene to Alexandria was equal to about 575 miles.

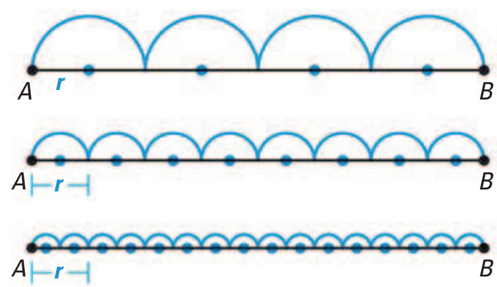
Find $m\angle 1$. Then estimate Earth's circumference.



CHALLENGE **Suppose $\overline{AB}$ is divided into four congruent segments, and semicircles with radius r are drawn.**

40. What is the sum of the four arc lengths if the radius of each arc is r?

41. Suppose that $\overline{AB}$ is divided into n congruent segments and that semicircles are drawn, as shown. What will the sum of the arc lengths be for 8 segments? for 16 segments? for n segments? *Explain* your thinking.



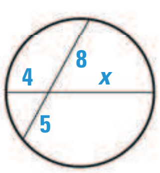
MIXED REVIEW

PREVIEW
Prepare for Lesson 11.5 in Exs. 42–45.

Find the area of a circle with radius r. Round to the nearest hundredth. *(p. 49)*

42. $r = 6$ cm 43. $r = 4.2$ in. 44. $r = 8\frac{3}{4}$ mi 45. $r = 1\frac{3}{8}$ in.

Find the value of x. *(p. 689)*

46.


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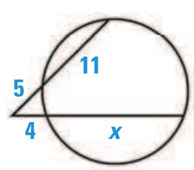

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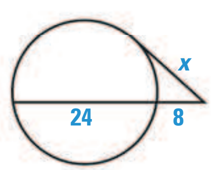


Extension

Use after Lesson 11.4

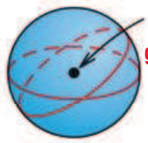
Geometry on a Sphere

GOAL Compare Euclidean and spherical geometries.

Key Vocabulary
- **great circle**

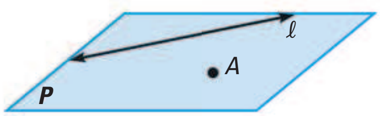
In Euclidean geometry, a plane is a flat surface that extends without end in all directions, and a line in the plane is a set of points that extends without end in two directions. Geometry on a sphere is different.

In *spherical geometry*, a plane is the surface of a sphere. A line is defined as a **great circle**, which is a circle on the sphere whose center is the center of the sphere.



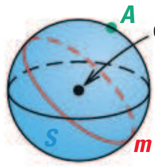
KEY CONCEPT — *For Your Notebook*

Euclidean Geometry



Plane P contains line ℓ and point A not on the line ℓ.

Spherical Geometry



Sphere S contains great circle m and point A not on m. Great circle m is a line.

HISTORY NOTE

Spherical geometry is sometimes called *Riemann geometry* after Bernhard Riemann, who wrote the first description of it in 1854.

Some properties and postulates in Euclidean geometry are true in spherical geometry. Others are not, or are true only under certain circumstances. For example, in Euclidean geometry, Postulate 5 states that through any two points there exists exactly one line. On a sphere, this postulate is true only for points that are not the endpoints of a diameter of the sphere.

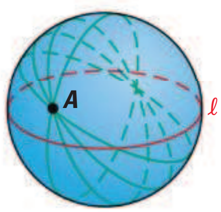
EXAMPLE 1 Compare Euclidean and spherical geometry

Tell whether the following postulate in Euclidean geometry is also true in spherical geometry. Draw a diagram to support your answer.

Parallel Postulate: If there is a line ℓ and a point A not on the line, then there is exactly one line through the point A parallel to the given line ℓ.

Solution

Parallel lines do not intersect. The sphere shows a line ℓ (a great circle) and a point A not on ℓ. Several lines are drawn through A. Each great circle containing A intersects ℓ. So, there can be no line parallel to ℓ. The parallel postulate is not true in spherical geometry.

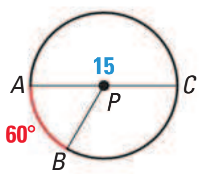


DISTANCES In Euclidean geometry, there is exactly one distance that can be measured between any two points. On a sphere, there are two distances that can be measured between two points. These distances are the lengths of the major and minor arcs of the great circle drawn through the points.

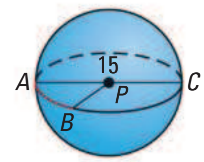
EXAMPLE 2 Find distances on a sphere

READ DIAGRAMS

The diagram below is a cross section of the sphere in Example 2. It shows $\widehat{AB}$ and $\widehat{ACB}$ on a great circle.



The diameter of the sphere shown is 15, and $m\widehat{AB} = 60°$. Find the distances between A and B.



Solution

Find the lengths of the minor arc $\widehat{AB}$ and the major arc $\widehat{ACB}$ of the great circle shown. In each case, let x be the arc length.

$$\frac{\text{Arc length of } \widehat{AB}}{2\pi r} = \frac{m\widehat{AB}}{360°} \qquad \frac{\text{Arc length of } \widehat{ACB}}{2\pi r} = \frac{m\widehat{ACB}}{360°}$$

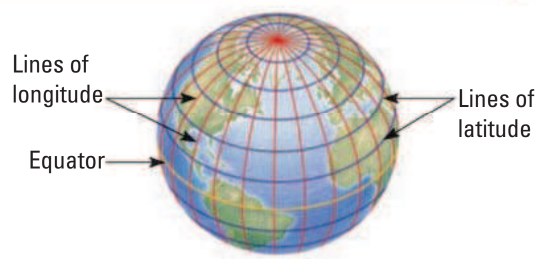
$$\frac{x}{15\pi} = \frac{60°}{360°} \qquad \frac{x}{15\pi} = \frac{360° - 60°}{360°}$$

$$x = 2.5\pi \qquad x = 12.5\pi$$

▶ The distances are 2.5π and 12.5π.

PRACTICE

1. **WRITING** Lines of latitude and longitude are used to identify positions on Earth. Which of the lines shown in the figure are great circles. Which are not? *Explain* your reasoning.



EXAMPLE 1 on p. 753 for Exs. 2–3

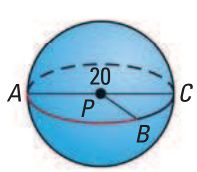
2. **COMPARING GEOMETRIES** Draw sketches to show that there is more than one line through the endpoints of a diameter of a sphere, but only one line through two points that are *not* endpoints of a diameter.

3. **COMPARING GEOMETRIES** The following statement is true in Euclidean geometry: If two lines intersect, then their intersection is exactly one point. Rewrite this statement to be true for lines on a sphere. *Explain*.

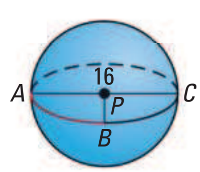
EXAMPLE 2 on p. 754 for Exs. 4–6

FINDING DISTANCES **Use the diagram and the given arc measure to find the distances between points A and B. Leave your answers in terms of π.**

4. $m\widehat{AB} = 120°$



5. $m\widehat{AB} = 90°$



6. $m\widehat{AB} = 140°$